Quantitative description of realistic wealth distributions by kinetic trading models

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Data on wealth distributions in trading markets show a power law behavior $x^{-(1+\alpha)}$ at the high end, where, in general, α is greater than 1 (Pareto's law). Models based on kinetic theory, where a set of interacting agents trade money, yield power law tails if agents are assigned a saving propensity. In this paper we are solving the inverse problem, that is, in finding the saving propensity distribution which yields a given wealth distribution for all wealth ranges. This is done explicitly for two recently published and comprehensive wealth datasets.

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Economic systems have received a great deal of attention from a physical viewpoint, sharing features common to a wide variety of complex systems [1]. Problems such as the emergence of a self-organized critical state [2,3], the predictability of financial markets [4] or stochastic decision making [5], have been studied in this framework.

In particular, the problem of wealth distribution in a society is ubiquitous and interesting, since abundant data is available. Certainly, it is complicated by the fact that a number of issues are relevant in the dynamics of wealth in a society, such as economic policies, natural resources, human psychology, competition, external markets, etc. A number of physics-based models have been developed, which, while simple, reproduce various features of economic systems.

Examples of these are the kinetic trading models, proposed by Refs. [6–8], that are represented by a number of agents, which interact by trading money. Total money is conserved and wealth is distributed across agents, eventually reaching an equilibrium distribution which depends on the details of the interaction. Thus, these models are analogous to a simulation of interacting particles in a gas where agents that trade and exchange money correspond to particles that collide and exchange energy. The analogy has proved to be extremely useful, and elements of Boltzmann transport theory have been used to study the evolution of these models [3,9]. For example, a standard simulation of a gas of hard spheres with elastic binary collisions has an exponential equilibrium distribution for the kinetic energy, denominated a Maxwell-Boltzmann distribution. Analogously, in the simplest version of kinetic trading models, agents may exchange any portion of their current wealth which, unsurprisingly, yields an exponential wealth distribution. However, in many cases (earthquake intensities, word frequencies in languages, citations of scientific papers, particle velocities in turbulent plasmas, etc.) power law distribution functions are observed, and economic systems are no exception [see Eq. (1) below]. Although general statistical mechanics frameworks have been proposed to deal with these systems (see, e.g., Refs. [10–12]), there is still the problem of identifying which features in a given dynamics are relevant to yield non-Maxwellian distribution functions.

In this paper we will concern ourselves specifically with the distribution of wealth. We show that a simple kinetic model for trading is able to reproduce observed wealth distribution data. The key factor is that each agent has a spending propensity, distributed nonuniformly across agents [2,6,7]. In this paper we solve the inverse problem of finding the spending propensity distribution which yields a given wealth distribution. By doing so, we show that kinetic trading models can fit observed data for all wealth ranges, not only in the high end tail where Pareto's law is valid. These findings also show that nondissipative binary interactions between otherwise independent agents, lead to power law distributions, as long as full exchange is not possible (see Ref. [13] for a similar suggestion). Such an insight, which has not been reported before as far as we know, may be useful to describe other physical systems which tend toward non-Maxwellian equilibria.

For economic systems, it is a well known observation that the probability of an agent of having wealth x is

$$P(x) \propto x^{-(1+\alpha)} \tag{1}$$

for large x, with an observed value of α between 1 and 2 [14,15]. This was first noted by Pareto in the 1890's [16], and has been also observed in different countries and in different periods of time [17–21]. As mentioned above, if agents can exchange any amount of their current wealth, a Maxwellian equilibrium is found. A power law tail can be obtained in more refined models, assigning a random saving propensity $0 \leq \lambda_i < 1$ to each agent, such that, at each interaction each agent only trades a certain amount of her or his wealth. However, these models yield power law tails with exponent 1 [2,3,9,22–24].

This difficulty can be overcome by choosing a nonuniform saving propensity distribution, but so far efforts have dealt with studying the effects of a particular choice [2,6,7,25], and to only fit the asymptotic behavior, in the Pareto regime. We will show, instead, that kinetic trading models are able to fit observed wealth distribution data not only for Pareto indexes $\alpha \neq 1$, and for all wealth ranges. Thus, these models can be quantitatively, not just qualitatively, consistent with observed data. In particular, we will show how to reproduce with our model two recently published data sets, one of them provided by the United Nations, and regarded as "the most comprehensive study of personal wealth ever undertaken" [26]. In turn, the model will provide us with the average spending propensity distribution, which may yield useful constraining information to more refined models.

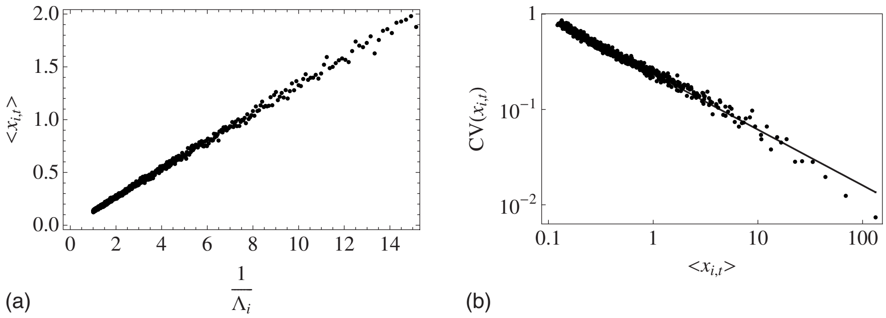

FIG. 1. (a) Scatter plot of $\langle x_{i,t}\rangle$ versus $1/\Lambda_i$. (b) Coefficient of variance of wealth versus average agents' wealth. Results from running a simulation of the exchange model (2), with $\Lambda_i=i/N$ (uniform distribution). The straight lines are a least squares fit of the data.

We start by reformulating these kinetic trading models in terms of an spending propensity $0<\Lambda_i\leqslant 1$ of each of the N agents. An agent i who has money $x_{i,t}$ at time t, exchanges part of her or his money with an agent j, such that at time $t+1$ their respective money is

$$x_{i,t+1} = (1-\Lambda_i)x_{i,t} + (\Lambda_i x_{i,t} + \Lambda_j x_{j,t})\epsilon_{i,j,t}, \tag{2}$$

$$x_{j,t+1} = (1-\Lambda_j)x_{j,t} + (\Lambda_i x_{i,t} + \Lambda_j x_{j,t})(1-\epsilon_{i,j,t}),$$

where self-interactions $j=i$ are not considered. Here t only labels time steps, and $\epsilon_{i,j,t}$ is taken from a uniform distribution $U(0,1)$. [Notice that there is no sum over repeated indexes in Eq. (2); whenever a sum is intended, it will be explicitly indicated.] Let us note that this two-agent exchange model conserves the total amount of money.

For $\Lambda_i=1$ (when agents can exchange any portion of the money they have), Eqs. (2) are analogous to the energy exchange equation between two particles in a standard simulation of a gas of hard spheres with elastic binary collisions. In this case, each collision must conserve total momentum $\vec{P}$ and kinetic energy E. For two particles i, j, if nonprimed letters represent quantities before the collision, and primed ones represent quantities after the collision, $E_{i,j}=E_i+E_j=E_i'+E_j'$. Since the direction of the velocities before the collision is in general random (for a random initial condition), the kinetic energy for each particle after the collision turns out to be a random quantity. Thus, $E_i'=r_{i,j}E_{i,j}=r_{i,j}(E_i+E_j)$, $E_j'=(1-r_{i,j})E_{i,j}=(1-r_{i,j})(E_i+E_j)$, with $r_{i,j}$ some random number between 0 and 1. In terms of energy, a collision simply consists of a random redistribution of the sum of the kinetic energies. This is the same as Eqs. (2), if $\Lambda_i=1$ and replacing energy E_i by money x_i. In this sense, the model described by Eqs. (2) is equivalent to a hard sphere particle simulation, seen only in "energy space." This observation suggests that we call this a "kinetic trading model" [6–8].

Equations (2) describe a single interaction at time t between two given agents i and j. The simulation is started by assigning to each agent a spending propensity $0<\Lambda_i\leqslant 1$, from a given distribution as discussed above, and a certain amount of money $x_{i,0}$. The initial conditions for the amount of money are not relevant for the equilibrium distribution, except for very singular situations. The model is then iterated for a long enough time, choosing at random which pair of agents interact at each time step. In every respect, the procedure is the same as if a saving propensity λ_i is used instead, as mentioned above. The change $\lambda_i\to\Lambda_i=1-\lambda_i$ may appear as trivial, but it turns out that high end tails are more sensitive to nonuniformity in Λ_i rather than λ_i [7]. This is due, in turn, to the nontrivial mapping of distributions when they are nonuniform. In the following, we will show how the equilibrium probability distribution of wealth is determined by the distribution of the Λ_i's. First, we can rewrite Eq. (2) as

$$x_{i,t+1} = (1-\Lambda_i)x_{i,t} + \epsilon_{i,j,t}(\Lambda_i x_{i,t} + w_{i,j,t}), \tag{3}$$

where $w_{i,j,t}=\Lambda_j x_{j,t}$, for $j\neq i$. We now average Eq. (3) over time. If $\epsilon_{i,j,t}$ is uncorrelated in time and with the choice of agents i and j, then this average can be written as

$$\langle x_{i,t+1}\rangle = \langle x_{i,t}\rangle - \langle\Lambda_i x_{i,t}\rangle + \langle\epsilon_{i,j,t}\rangle(\langle\Lambda_i x_{i,t}\rangle + \langle w_{i,j,t}\rangle). \tag{4}$$

For long enough times, agent i will interact with every other agent j several times, so that $\langle w_{i,j,t}\rangle$ can be regarded as an average both over agents and time. Thus, it makes sense to calculate it as an average over j, followed by an average over t:

$$\langle w_{i,j,t}\rangle = \left\langle \frac{1}{N-1}\sum_{j=1,j\neq i}^{N}\Lambda_j x_{j,t}\right\rangle \xrightarrow[N\to\infty]{} \frac{\sum_{j=1}^{N}\langle\Lambda_j x_{j,t}\rangle}{N}. \tag{5}$$

Taking the average of $\epsilon_{i,j,t}$ as $1/2$, and noting that the equilibrium distribution is stationary, i.e., $\langle x_{i,t+1}\rangle=\langle x_{i,t}\rangle$, Eq. (4) yields

$$\langle\Lambda_i x_{i,t}\rangle = \frac{\sum_{j=1}^{N}\langle\Lambda_j x_{j,t}\rangle}{N}. \tag{6}$$

Equation (6) shows that the random variable $\langle\Lambda_i x_{i,t}\rangle$ is equal to its mean for all i, therefore it is a constant $\langle\Lambda_i x_{i,t}\rangle=\kappa$. Since we have taken Λ_i as constant in time, we have

$$\langle x_{i,t}\rangle = \frac{\kappa}{\Lambda_i}. \tag{7}$$

Figure 1(a) illustrates Eq. (7), presenting results from running a simulation with model (2), with Λ_i taken from a uniform distribution ($\Lambda_i=i/N$). κ can be calculated using the conservation of money $\Sigma_{i=1}^{N}\langle x_{i,t}\rangle=M$, where M is the total amount of money in the system.

Using Eq. (7) we may establish a connection between the wealth distribution $P(x)$ and the spending propensities distribution $P(\Lambda)$. First, we notice that in the relevant wealth range $\langle x_{i,t}\rangle\simeq x_i$, since dispersion is small [see Fig. 1(b)]. Thus, since $P(x)dx=P(\Lambda)d\Lambda$, we obtain

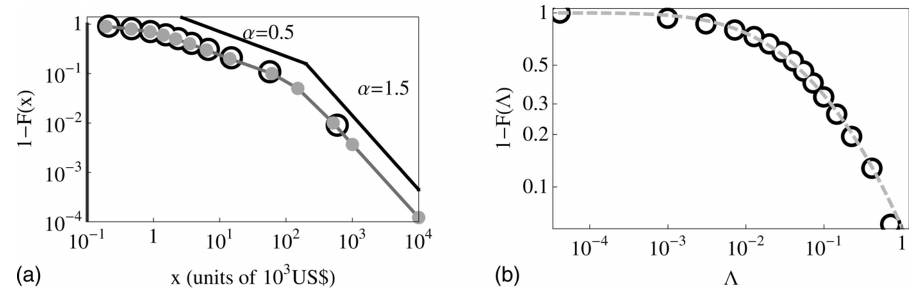

FIG. 2. Data and model fit for UNU-WIDER's data [26]. Rings, model results; dots, original data; light line, analytic results from Eq. (7); dark line, power-law fit. (a) Distribution function for wealth x, from UNU-WIDER's data. (b) Distribution function for spending propensity Λ, fitted by a log-normal distribution (dashed line). $F(x)$ and $F(\Lambda)$ denote cumulative distribution functions.

$$P(x) = P(\Lambda)|_{\Lambda=\kappa/x}\frac{\kappa}{x^2}. \tag{8}$$

Equations (7) and (8) are consistent with Refs. [6,7]. For a uniform distribution of Λ, $P(\Lambda)$ constant, we recover the known result that the wealth distribution follows a power law with Pareto index $\alpha=1$ [2,3,9,22–24].

Equations (7) and (8) establish how the distribution in spending propensities $P(\Lambda)$ determines the wealth distribution $P(x)$. They show that, by using a proper distribution for Λ_i, in principle one should be able to reproduce any power law behavior at the high end of the wealth distribution, not just $\alpha=1$. If $P(\Lambda)\simeq\Lambda^{\beta}$ for small values of Λ, then the Pareto index is $1+\beta$, an asymptotic result also found in Refs. [2,6,7]. However, we will show that they provide a systematic way to solve the inverse problem, of finding the spending propensity distribution $P(\Lambda)$ which yields a given wealth distribution. In doing so, we will also show that kinetic trading models can fit observed distributions for all wealth ranges, not only at the high end. This will be explicitly done by simulating two recently published wealth distribution datasets.

In 2006, UNU-WIDER published the World Distribution of Household Wealth, being the most complete study of personal wealth currently available, including datasets for income distribution, financial assets and debts, land, buildings, etc., and covering 228 countries [26]. This study presents data on purchasing power parity (PPP) and official exchange rate bases. For this work, we have considered the latter, which is preferable when studying large values of wealth [26]. However, the main conclusions do not change if PPP data are used.

Data for this study are plotted as gray dots in Fig. 2(a). Two regimes of power law behavior can be observed. The value of α, calculated by the Newman's method [27] and the Kolmogorov-Smirnov's test [28], is $\alpha\simeq1.5$ in the large wealth range, consistent with Pareto's findings [16] and other studies [14,15,19–21].

We now attempt to simulate this data using model (2). In order to do that, we need an appropriate distribution of Λ_i. First, we calculate the cumulative distribution function of wealth data $F(x)$ from UNU-WIDER's data. Then, from Eq. (7) it follows that $u=F(\Lambda)\propto1/F(x)$, so that we can construct $\Lambda_i=F^{-1}(u_i)$ for N uniform random numbers, with $0\leqslant u_i\leqslant1$. Alternatively, we may set $u_i=i/N$ [22]. Using this set of Λ_i we run the model (2) and look at the resultant equilibrium wealth distribution. Results are presented in Figs. 2(a) and 2(b) for wealth and Λ distributions, respectively. It can be seen that this simple model of exchange with spending propensities is able to reproduce the observed data, even if the Pareto index is clearly different from 1. Furthermore, the model works over the whole range of wealth, not just the high-end tail.

In order to consider a better data source for large wealth values, we also analyzed the 2006 Forbes list of billionaires of the world. Data are plotted as gray dots in Fig. 3(a), as well as their fit by a power law distribution with $\alpha=1.4$, close to the original observation by Pareto [16], and the results of the UNU-WIDER data. Hence they represent two consistent data sets, and complement each other. Using the procedure outlined above, we reconstructed the billionaires wealth distribution using the model equation (2). Figure 3(a) shows that the model generates a wealth distribution that is in very good agreement with Forbes's data, once the proper spending propensity distribution is determined [Fig. 3(b)].

For both datasets, UNU-WIDER and Forbes, a spending propensity distribution $F(\Lambda)$ has been obtained, consistent with the corresponding wealth distribution. However, it is interesting to note that, within its more limited range,

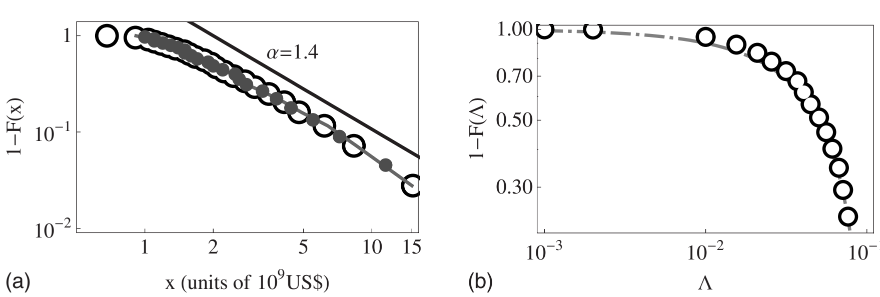

FIG. 3. Same as Fig. 2, but for Forbes's world billionaires wealth distribution. (a) Forbes's world billionaires wealth distribution. (b) Λ distribution for Forbes's data. It slightly departs from a uniform distribution (dashed line).

Forbes's data yield a distribution which is consistent with UNU-WIDER's data, both for $F(x)$ and $F(\Lambda)$.

In summary, we have studied a simple trading model based on the kinetic theory of gases. When individual spending propensities are introduced in these models, it is shown that the model is able to reproduce observed wealth equilibrium distribution data, not only in the asymptotic regime, with Pareto index different than 1, but also for all wealth ranges. This is done explicitly by solving the inverse problem of finding the spending propensity distribution given two recent wealth distribution estimates, and then running the model. The tail of the wealth distribution is more sensitive to nonuniformity in the distribution of Λ_i rather than $\lambda_i = 1 - \Lambda_i$. The strong correlation between wealth x and the inverse of spending propensity Λ_i, as expressed in Eq. (7), implies that a given variation $\Delta\Lambda/\Lambda_i$ has a larger effect if the individual's spending propensity Λ_i is closer to zero than to one. This means, for instance, that decreasing the consumption level of a poor individual, which in itself implies a big sacrifice on her/his part since saving capabilities are lower, has less incidence on the overall wealth distribution than increasing the consumption level of a rich individual. A result which may at first sight seem obvious, but the strength of the model is that we can quantitatively estimate how efficient this effect is, and thus can have interesting consequences in terms of economic policies development.

We would like to stress here that the model provides two distributions, a wealth distribution which reproduces observed data, and a spending propensity distribution. In this sense, we can regard the Λ distribution as a prediction of the model, which can, in principle, be tested against real data. However, as far as we know, such data is not readily available, at least for the world datasets we analyzed. Certainly, it would be interesting to validate the Λ distribution by independent means, but that requires to have reliable data for a given community, both for its spending propensity and wealth. We plan to pursue such line of research in a future work.

Finally, it is interesting to note that traditionally, the non-Maxwellian distribution functions have been related to other features, such as correlated systems, memory effects or dissipative interactions [10]. However, the model studied here has nondissipative binary interactions between otherwise independent agents. It differs from a regular gas with elastic interactions only in the fact that full exchange between agents is not permitted (while still being conservative). Yet, this seemingly naive change in the dynamics can have a profound effect on the equilibrium distribution function [13], and provides an insight which may be useful for other physical systems exhibiting equilibrium distributions with non-Maxwellian behavior, where similar underlying dynamics may be involved.

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