Relativistic mass and charge of photons in thermal plasmas through electromagnetic field quantization

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An effective photon mass and equivalent photon charge are calculated for plasmas with finite temperature, by using a second covariant quantization of the electromagnetic field, which is based on a nonlinear magnetofluid unification field formalism. Relativistic effects are considered both in the fluid bulk motion and in the thermal motion. The effective relativistic photon mass is found for transverse and longitudinal photons, while the equivalent relativistic photon charge is obtained for purely transverse photons. Both quantum quantities are the relativistic generalization, at finite temperature, of previous results [Mendonça *et al.*, Phys. Rev. E **62**, 2989 (2000)]. The dependence with temperature is studied in both cases.

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For nonrelativistic cold plasmas, the equation for electromagnetic wave propagation is analogous to the Klein-Gordon equation for massive vector fields, which allows us to calculate an effective photon mass. This effective mass is proportional to the plasma frequency and it is thus a linear effect. Similarly, a photon mass can be obtained in a quantum field scheme by considering a symmetry breaking mechanism [1]. In addition to the effective photon mass, it is possible to associate an electric charge to the photon in cold nonrelativistic plasmas. This equivalent charge is, in fact, a nonlinear effect related to the ponderomotive force. Furthermore, the concept of effective mass and charge have been extended to neutrinos in plasmas [2–5].

More recently, Mendonça *et al.* [6] showed that it is possible to find an effective photon mass and an equivalent photon charge when the canonical formalism of second quantization is applied to cold nonrelativistic plasmas. Even though it is not applied to a quantum system, the second field quantization formalism provides a standard methodology to analyze the complete solution, including the nonlinear terms, and allows us to interpret these effects as "quantizations" of the photons of the plasma wave modes, providing a deeper meaning than in the classical interpretation. In this paper, following the procedure in Ref. [6], we apply the second quantization formalism to obtain the effective mass and the equivalent charge of photons in a relativistic plasma with temperature. In order to include thermal effects, the magnetofluid unification field formalism proposed in Ref. [7] is used. In this formalism, the electromagnetic field and a charged fluid are unified in a single field, which has its own equation of motion. Thus, the plasma can be described by only one field variable, which is specially suitable for a second quantization approach since it considers the relativistic thermal fluid and the electromagnetic fields in the same footing. This fluid theory is fully relativistic, both for the plasma bulk and thermal motion.

Following Ref. [7], we start with the set of equations that describe the high frequency behavior of a thermal relativistic plasma. The electromagnetic field dynamics is described by Maxwell equations

$$F^{\nu\mu}_{\ ,\mu} = -4\pi J^{\nu}, \tag{1}$$

where $F^{\mu\nu}=\partial^{\mu}A^{\nu}-\partial^{\nu}A^{\mu}$ are the components of the electromagnetic tensor, $A^{\mu}\rightarrow(\phi,\mathbf{A})$ is the four-vector potential, ϕ is the scalar potential, and $\mathbf{A}$ is the vector potential. If α labels particle species (electrons and ions), the total four-current is $J^{\mu}=\Sigma_{\alpha}J^{\mu}_{(\alpha)}$, where $J^{\nu}_{(\alpha)}=q_{(\alpha)}n_{(\alpha)}U^{\nu}_{(\alpha)}$, n is the electron density in the rest frame, $U^{\mu}_{(\alpha)}\rightarrow(\gamma_{(\alpha)},\mathbf{U}_{(\alpha)})=(\gamma_{(\alpha)},\gamma_{(\alpha)}\mathbf{v}_{(\alpha)})$ is the four velocity of the fluid, $\mathbf{v}_{(\alpha)}$ is the velocity, and $\gamma_{(\alpha)}=(1-|\mathbf{v}_{(\alpha)}|^2)^{-1/2}$. (In this paper, we have taken the speed of light $c=1$, and Boltzmann constant $k_B=1$.) Besides, Eq. (1) gives the continuity equation for each species

$$\partial_{\nu}J^{\nu}_{(\alpha)}=0. \tag{2}$$

Dropping the species index, the equation of motion for electrons ($q=-e$) with mass m is

$$\partial_{\nu}T^{\mu\nu}=F^{\mu\nu}J_{\nu}, \tag{3}$$

where the energy-momentum tensor for a relativistic fluid is given by [8]

$$T^{\mu\nu}=p\,\eta^{\mu\nu}+hU^{\mu}U^{\nu}. \tag{4}$$

Here p is the electron scalar pressure, $\eta^{\mu\nu}\rightarrow\mathrm{diag}(-1,1,1,1)$ is the signature, and h is the enthalpy density. For a noninteracting relativistic electron gas, $h=mnf(T)$, where f is a function depending only on the temperature T, namely [7]

$$f(T)=\frac{K_3\left(\frac{m}{T}\right)}{K_2\left(\frac{m}{T}\right)}. \tag{5}$$

Here, K_3 and K_2 are the modified Bessel functions of order 3 and 2, respectively. A derivation of this enthalpy can be found in the Appendix.

It was noticed in Ref. [7] that it is possible to rewrite Eq. (3) for the motion of electrons as

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$$U_\mu M^{\mu\nu} = 0, \tag{6}$$

where $M^{\mu\nu}$ is a new tensor that couples the electromagnetic field A^μ and the fluid field U^μ in the form $M^{\mu\nu}=F^{\mu\nu}+(m/q)[\partial^\mu(fU^\nu)-\partial^\nu(fU^\mu)]$. Thus, Eq. (6) describes a nondegenerate charged thermal relativistic plasma, where all thermal information is contained in the function $f(T)$.

The spacelike components of Eq. (6) for electrons can be written as

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\right)(f\gamma\mathbf{v}) = \frac{q}{m}(\mathbf{E}+\mathbf{v}\times\mathbf{B}) - \frac{1}{mn\gamma}\boldsymbol{\nabla} p, \tag{7}$$

where $\mathbf{E}=-\boldsymbol{\nabla}\phi-\partial_t\mathbf{A}$ is the electric field and $\mathbf{B}=\boldsymbol{\nabla}\times\mathbf{A}$ is the magnetic field. It is possible to show that Eq. (7) is equivalent to the well-known relativistic fluid equation of charged particles [9]. Furthermore, in the homentropic regime and in the low-temperature nonrelativistic limit, Eq. (6) reduces to the standard nonrelativistic charged fluid equation [7].

I. EFFECTIVE RELATIVISTIC PHOTON MASS

We now show that when the second quantization formalism is used to quantize the electromagnetic field in a plasma under the previous scheme, it is possible to associate a massive vector field to the photon field.

First, we will linearize the equation of motion for the electromagnetic and charged fluid fields with respect to an equilibrium state consisting of a plasma moving with a constant longitudinal velocity $\mathbf{v}_0=v_0\hat{z}$ and null electromagnetic fields. The ions will be a fixed background. For the sake of simplicity, we will assume that every perturbed quantity has a space and time dependence given by $\exp(ikz-i\omega t)$. In this case, the perturbed scalar electrostatic potential ϕ is longitudinal, i.e., its gradient has only components in the longitudinal direction $\hat{z}$. We can write the density as $n=n_0+n_1$, where n_0 is the equilibrium electron density in the rest frame (same for ions), and n_1 is a first order electron density perturbation.

In the Lorentz gauge, Eq. (1) yields the wave equation

$$\partial_\mu\partial^\mu A^\nu = 4\pi J^\nu. \tag{8}$$

The linearization of this equation and Eq. (7) yields the longitudinal (electrostatic) and the transverse (electromagnetic) modes.

First, we focus on the longitudinal modes. For these modes, the velocity of the electrons is $\mathbf{v}=\mathbf{v}_0+\mathbf{v}_1$, where $\mathbf{v}_1=v_1\hat{z}$ is the longitudinal velocity perturbation in response to the potential ϕ. The relativistic factor will be $\gamma=\gamma_0+\gamma_1$, where $\gamma_0=(1-v_0^2)^{-1/2}$ is constant, and $\gamma_1=\gamma_0^3 v_0 v_1$ is the first order perturbation.

In this case, Eq. (8) can be written for the scalar potential ϕ as

$$\partial_\mu\partial^\mu\phi = 4\pi q n^L, \tag{9}$$

where $n^L=\gamma_0 n_1+\gamma_1 n_0$ is the electron density perturbation in the laboratory frame.

The continuity equation (2), at first order in the perturbation quantities, is

$$\frac{\partial n^L}{\partial t} + \boldsymbol{\nabla}\cdot(\gamma_0 n_0\mathbf{v}_1 + n^L\mathbf{v}_0) = 0, \tag{10}$$

whereas the first order longitudinal perturbation for Eq. (7) yields

$$\left(\frac{\partial}{\partial t}+v_0\frac{\partial}{\partial z}\right)(\gamma_0\mathbf{v}_1+\gamma_1\mathbf{v}_0) = \frac{q\mathbf{E}_1}{mf} - \frac{1}{mn_0\gamma_0 f}\boldsymbol{\nabla} p - \frac{1}{f}\left(\frac{\partial f}{\partial t}+v_0\frac{\partial f}{\partial z}\right)(\gamma_1\mathbf{v}_0+\gamma_0\mathbf{v}_1) - \frac{v_1}{f}\frac{\partial f}{\partial z}\gamma_0\mathbf{v}_0. \tag{11}$$

$\mathbf{E}_1$ is the electric field at first order perturbation produced by the longitudinal perturbed quantities. The pressure p, according to Eq. (A7), is $p=(n_0+n_1)T$ [10], so that the pressure fluctuations can be written as $\boldsymbol{\nabla}p=\frac{\partial p}{\partial n_1}\boldsymbol{\nabla}n_1=mv_e^2\boldsymbol{\nabla}n_1$, where $v_e=\sqrt{T/m}$ is the electron thermal velocity.

Deriving Eq. (10), and using Eq. (11) we have

$$\frac{1}{n_0}\left(\frac{\partial}{\partial t}+v_0\frac{\partial}{\partial z}\right)^2 n_L + \boldsymbol{\nabla}\cdot\left(\frac{q\mathbf{E}_1}{mf} - \frac{v_e^2}{n_0\gamma_0^2 f}\boldsymbol{\nabla} n^L + \frac{v_e^2}{f\gamma_0^2}\boldsymbol{\nabla}\gamma_1\right) - \left(\frac{\partial}{\partial t}+v_0\frac{\partial}{\partial z}\right)v_0\frac{\partial\gamma_1}{\partial z} - \frac{\partial}{\partial z}\left[\frac{\gamma_0 v_0 v_1}{f}(1+\gamma_0^2)\frac{\partial f}{\partial z}\right] - \frac{\partial}{\partial z}\left[\frac{1}{f}\frac{\partial f}{\partial t}(\gamma_0 v_1+\gamma_1 v_0)\right] = 0. \tag{12}$$

Now, we will focus on a constant temperature plasma, where the f function does not depend on time nor space. Thus, all first order perturbations are due to the electromagnetic fields. We also assume that plasma oscillations are isothermal.

In the Lorenz gauge $\boldsymbol{\nabla}\cdot\mathbf{E}_1=\partial_\mu\partial^\mu\phi=4\pi q n^L$. Using this and solving Eq. (10) for v_1 to find γ_1, it is possible to show that Eq. (12) yields the wave equation for electrostatic modes

$$\left[\left(\frac{\partial}{\partial t}+v_0\frac{\partial}{\partial z}\right)^2 - \frac{v_e^2}{f\gamma_0^4}\frac{\partial^2}{\partial z^2}\right]\phi = -\frac{\omega_p^2}{f\gamma_0^2}\phi + \frac{v_0 v_e^2}{f\gamma_0^2}\frac{\partial}{\partial z}\left(\frac{\partial}{\partial t}+v_0\frac{\partial}{\partial z}\right)\phi, \tag{13}$$

where $\omega_p=\sqrt{4\pi e^2 n_0/m}$ is the background electron plasma frequency.

Now, we turn our analysis to transverse modes. Here, the velocity is $\mathbf{v}=\mathbf{v}_0+\mathbf{v}_\perp$, where $\mathbf{v}_\perp$ is a transverse velocity perturbation such that $\mathbf{v}_\perp\cdot\mathbf{v}_0=0$. Then, the relativistic factor is the constant γ_0 to first order. The corresponding Eq. (8) for the transverse part of the vector potential $\mathbf{A}_\perp$ is

$$\partial_\mu\partial^\mu\mathbf{A}_\perp = 4\pi q n_0\mathbf{U}_\perp, \tag{14}$$

where $\mathbf{U}_\perp=\gamma_0\mathbf{v}_\perp$. Furthermore, the transverse part of the equation of motion (7) can be written

$$\frac{d}{dt}(f\mathbf{U}_\perp) = -\frac{q}{m}\frac{d}{dt}(\mathbf{A}_\perp). \tag{15}$$

Hence, we have

$$f\mathbf{U}_\perp = -\frac{q}{m}\mathbf{A}_\perp, \tag{16}$$

and then, using Eq. (16) in Eq. (14), we obtain the wave equation for transverse modes

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\mathbf{A}_\perp = -\frac{\omega_p^2}{f}\mathbf{A}_\perp. \tag{17}$$

Now that we have wave equations for ϕ [Eq. (13)] and $\mathbf{A}_\perp$ [Eq. (17)], we can use covariant second quantization for the electromagnetic field through the Gupta-Bleuler formalism in the Lorentz gauge [11]. The A^μ potential can be expanded as a Fourier decomposition of creation and annihilation operators as follows:

$$A^\mu(z,t) = A^{\mu+}(z,t) + A^{\mu-}(z,t), \tag{18}$$

where

$$A^{\mu+}(z,t) = \sum_k \sum_{r=0}^{3} \sqrt{\frac{\hbar}{2\omega}}\epsilon_r^\mu(k) a_r(k) e^{-i(kz-\omega t)}, \tag{19}$$

and

$$A^{\mu-}(z,t) = \sum_k \sum_{r=0}^{3} \sqrt{\frac{\hbar}{2\omega}}\epsilon_r^\mu(k) a_r^\dagger(k) e^{i(kz-\omega t)}, \tag{20}$$

where k is the wave number, $\omega(k) \equiv \omega$ is the frequency, and $\hbar$ is the reduced Planck constant. In covariant quantization, there exist, for each k, four linearly independent polarization states for the four-vector A^μ which are represented in our case by the polarization vectors $\epsilon_0^\mu(k) \to (1,0,0,0)$, $\epsilon_1^\mu(k) \to (0,1,0,0)$, $\epsilon_2^\mu(k) \to (0,0,1,0)$, and $\epsilon_3^\mu(k) \to (0,0,0,1)$. Thus, in Eqs. (19) and (20), r labels photon polarizations: $r=1,2$ for transverse photons, and $r=3$ for longitudinal photons ($r=0$ is called scalar polarization) [11]. Besides, note that due to the Lorentz gauge, A^0 and A^3 are not linearly independent. On the other hand, the operators of creation and annihilation satisfy the commutator relations $[a_r(k), a_p^\dagger(k')] = \xi_r \delta_{rp} \delta_{k,k'}$, where $\xi_r = 1$ for $r=1,2,3$ and $\xi_0 = -1$. Other commutators vanish.

Considering expansion (18) in Eq. (17) we can write the dispersion relation for transverse photons as

$$\omega_r^2 = k^2 + \frac{\omega_p^2}{f}, \tag{21}$$

with $r=1,2$. In the same token, expansion (18) in Eq. (13) gives the dispersion relation for longitudinal photons, or plasmons, as (with subindex 3)

$$(\omega_3 - v_0 k)^2 - \frac{v_e^2}{f\gamma_0^4}k^2 = \frac{\omega_p^2}{f\gamma_0^2} - \frac{v_0 v_e^2}{f\gamma_0^2}k(\omega_3 - v_0 k). \tag{22}$$

Equation (21) is equivalent to the dispersion relation of a relativistic free massive scalar field satisfying a quantized Klein-Gordon equation [11]. In analogy with this theory, we can define an effective mass for transverse photons as

$$M_{1,2}(T) = \frac{\hbar\omega_p}{\sqrt{f(T)}}. \tag{23}$$

On the other hand, Eq. (22) represents the dispersion relation for a quantized Klein-Gordon field where the operators are rotated in ω-k space. The effective mass for longitudinal photons is given by

$$M_3(T) = \frac{\hbar\omega_p}{c_3^2\gamma_0\sqrt{f(T)}}, \tag{24}$$

where the velocity c_3 is defined as the phase velocity when k is large, *i.e.*, $c_3 = \lim_{k\to\infty} \omega_3/k$. Then,

$$c_3 = v_0 + \frac{v_e}{2f\gamma_0^2}(\sqrt{v_0^2 v_e^2 + 4f} - v_0 v_e). \tag{25}$$

Let us note that the methodology we use to compute the phase velocity c_3 is also consistent with the transverse phase velocities used in Eq. (21) since in this case $\lim_{k\to\infty}\omega_{1,2}/k = 1$, as it should be.

The effective photon mass found for transverse and longitudinal modes is fully relativistic, and it arises as a response to the interaction of the electromagnetic wave modes and the collective behavior of the thermal plasma. Its dependence on temperature is through the $f(T)$ function given by Eq. (5), and the thermal velocity v_e. These masses can be thought of as the oscillations of the quantized electromagnetic wave equations.

Let us examine the temperature dependence of the effective mass. For low temperatures $f(T \ll m) \approx 1 + 5T/(2m)$, and hence, the effective transverse photon mass is

$$M_{1,2}(T \ll m) = \hbar\omega_p\left(1 - \frac{5T}{4m}\right), \tag{26}$$

whereas the effective longitudinal photon mass is

$$M_3(T \ll m) = \frac{\hbar\omega_p}{\gamma_0}\frac{\left(1 - \dfrac{5T}{4m}\right)}{\left\{v_0 + \dfrac{v_e}{\gamma_0^2}\left[1 - \dfrac{5T}{4m} + \dfrac{v_0^2 T}{8m} - \dfrac{v_0 v_e}{2}\right]\right\}^2}. \tag{27}$$

In the non-relativistic limit ($v_0 \to 0, \gamma_0 \to 1$) and low temperature regime ($T \to 0$), the effective mass for transverse photons is $M_{1,2} = \hbar\omega_p$, and the effective mass for plasmons becomes $M_3 = m\hbar\omega_p/T$, in agreement with Ref. [6]. On the other hand, in the high temperature limit, $f(T \gg m) \approx 4T/m$, the effective transverse photon mass is

$$M_{1,2}(T \gg m) = \frac{\hbar\omega_p}{2}\sqrt{\frac{m}{T}}, \tag{28}$$

and the effective longitudinal photon mass is

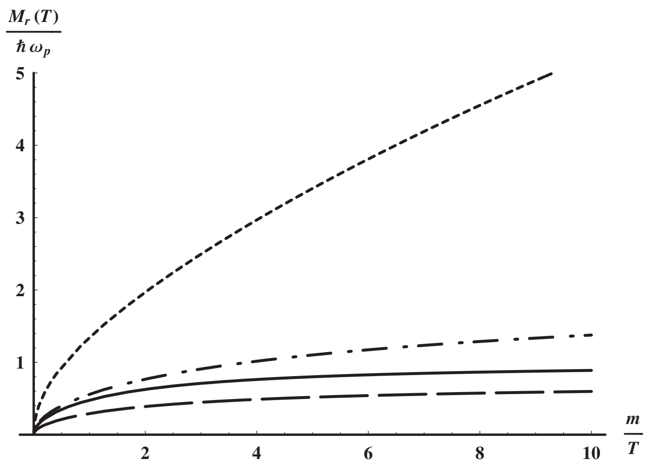

FIG. 1. Temperature dependence of the photon mass (23) and (24), with respect to m/T. The full line represents the mass of transverse photons ($r=1,2$). The other dashed lines correspond to the mass of longitudinal photons ($r=3$). The large-dashed line is for $\gamma=1.8$, the dot-dashed line is for $\gamma=1.2$, and the dashed line is for $\gamma=1.01$.

$$M_3(T \gg m) = \frac{\hbar\omega_p\sqrt{m/T}}{2\gamma_0[v_0 + (\sqrt{v_0^2+16} - v_0)/(8\gamma_0^2)]^2}. \quad (29)$$

Figure 1 shows the temperature dependence of the relativistic photon mass for transverse photon modes [Eq. (23)] (full line) and for longitudinal photon modes [Eq. (24)] (dashed lines) as a function of m/T. Notice how the mass of the longitudinal photons decreases when the relativistic factor increases.

Eventually, this approximation will break down, though. For an ultrarelativistic plasma, where electrons and photons are in thermodynamical equilibrium, $n \sim T^3$ [12], affecting the plasma frequency. This leads to an increasing photon mass at very large temperatures, $M \sim \omega_p/\sqrt{T} \sim \sqrt{n/T} \sim T$, in agreement with previous results [13,14].

II. EQUIVALENT RELATIVISTIC PHOTON CHARGE

We can also associate an equivalent charge to the photon through the second quantization formalism, as shown for a nonrelativistic cold plasma in Ref. [6]. Let us first notice that a transverse electromagnetic field will induce a nonlinear relativistic ponderomotive force [15]. In the context of a second quantization approach, the ponderomotive force can be reinterpreted as due to a charged photon repelling the electrons. Thus, when nonlinear density perturbations are considered, the photon acquires an effective negative charge.

On the same initial equilibrium as in Sec. I, we introduce for the purpose of illustration a nonlinear, second order, density perturbation n_2, and a corresponding second order velocity perturbation $\mathbf{v}_2$ so that $n=n_0+n_1+n_2$ and $\mathbf{v}=\mathbf{v}_0+\mathbf{v}_1+\mathbf{v}_2$. Here $\mathbf{v}_0=v_0\hat{z}$ is the constant longitudinal bulk velocity, $\mathbf{v}_1$ is the transverse velocity perturbation ($\mathbf{v}_0\cdot\mathbf{v}_1=0$), and $\mathbf{v}_2=v_2\hat{z}$ is a longitudinal velocity perturbation. In this way, the relativistic factor is $\gamma=\gamma_0+\gamma_2$, where the second order correction to the relativistic factor is

$$\gamma_2 = \gamma_0^3\left(v_0v_2 + \frac{v_1^2}{2}\right), \quad (30)$$

with $v_1=|\mathbf{v}_1|$.

The electric field can be written as $\mathbf{E}=\mathbf{E}_1+\mathbf{E}_2$, where $\mathbf{E}_1$ is the first order electric field, and $\mathbf{E}_2$ is the second order nonlinear field [4] ($|\mathbf{E}_1| \gg |\mathbf{E}_2|$). We make the same assumptions on temperature as before, namely, f is constant in time and space. We then seek solutions of the dynamical variables with only z and t dependence.

Now, at second order, the continuity equation (2) yields the equation for n_2

$$\frac{\partial n_2^L}{\partial t} + \nabla\cdot(\gamma_0 n_0 \mathbf{v}_2 + n_2^L\mathbf{v}_0) = 0, \quad (31)$$

where $n_2^L=\gamma_0 n_2+\gamma_2 n_0$ is the second-order density perturbation in the laboratory frame. To first order, Eq. (7) yields

$$\left(\frac{\partial}{\partial t} + v_0\frac{\partial}{\partial z}\right)(f\gamma_0\mathbf{v}_1) = \frac{q}{m}(\mathbf{E}_1 + \mathbf{v}_0\times\mathbf{B}_1), \quad (32)$$

as there is no background magnetic field. An expression similar to Eq. (16) can also be obtained,

$$f\gamma_0\mathbf{v}_1 = -\frac{q}{m}\mathbf{A}_\perp. \quad (33)$$

Similarly, given that the term $\mathbf{v}_0\times\mathbf{B}_2$ does not have longitudinal components, the relevant second order equation of motion can be written as

$$\left(\frac{\partial}{\partial t} + v_0\frac{\partial}{\partial z}\right)(\gamma_0\mathbf{v}_2 + \gamma_2\mathbf{v}_0) = \frac{q}{mf}(\mathbf{E}_2 + \mathbf{v}_1\times\mathbf{B}_1). \quad (34)$$

Just as before, we restrict ourselves to isothermal systems . In the above equation we do not consider nonlinear perturbations in the pressure because we are only focusing on the ponderomotive force due to the electromagnetic fields. Thus, we neglect the second order pressure terms $v_e^2 n_1\nabla n_1/(\gamma_0 n_0^2)+v_e^2\nabla\gamma_2/\gamma_0^2-v_e^2\nabla n_2^L/(n_0\gamma_0^2)$. This allows us to obtain a result that can be compared with a previous description of photon charge [6] in the cold nonrelativistic limit.

Using Eq. (33), Eq. (34) can be rewritten as

$$\left(\frac{\partial}{\partial t} + v_0\frac{\partial}{\partial z}\right)(\gamma_0\mathbf{v}_2 + \gamma_2\mathbf{v}_0) = \frac{q}{mf}\mathbf{E}_2 - \frac{q^2}{2m^2f^2\gamma_0}\nabla|\mathbf{A}_\perp|^2. \quad (35)$$

Combining Eqs. (31) and (35), and using $\nabla\cdot\mathbf{E}_2=4\pi q n_2^L$, we have

$$\left(\frac{\partial}{\partial t} + v_0\frac{\partial}{\partial z}\right)^2 n_2^L + \frac{\omega_p^2}{f}n_2^L = \frac{n_0q^2}{2m^2f^2\gamma_0}\frac{\partial^2}{\partial z^2}|\mathbf{A}_\perp|^2 + n_0v_0\frac{\partial}{\partial z}\left(\frac{\partial}{\partial t} + v_0\frac{\partial}{\partial z}\right)\gamma_2. \quad (36)$$

Solving Eq. (35) for $\gamma_0 v_2$, and then calculating γ_2 from Eq. (30), we obtain an evolution equation for the density perturbation n_2^L as a response to the relativistic ponderomotive force [15], namely,

$$\left(\frac{\partial}{\partial t}+v_0\frac{\partial}{\partial z}\right)^2 n_2^L+\frac{\omega_p^2}{f\gamma_0^2}n_2^L = \frac{n_0 q^2}{2m^2f^2\gamma_0}\frac{\partial^2}{\partial z^2}|\mathbf{A}_\perp|^2+\frac{n_0v_0q^2}{2m^2f^2\gamma_0}\frac{\partial}{\partial t}\frac{\partial}{\partial z}|\mathbf{A}_\perp|^2. \quad (37)$$

The nonlinear perturbation n_2^L is dominated by the ponderomotive force effect, which pushes the electrons away from the transverse field. We now consider a transverse wave packet which moves with no significant deformation through the plasma with group velocity v_r, in which case $\partial_z|\mathbf{A}_\perp|^2 \approx (1/v_r)\partial_t|\mathbf{A}_\perp|^2$ [6]. Here, $v_r=\partial\omega_r/\partial k$,where ω_r is the frequency of the transverse electromagnetic wave ($r=1,2$) as given by Eq. (21). Then, Eq. (37) becomes

$$\left(\frac{\partial}{\partial t}+v_0\frac{\partial}{\partial z}\right)^2 n_2^L+\frac{\omega_p^2}{f\gamma_0^2}n_2^L=\frac{\omega_p^2(1+v_0v_r)}{8\pi m v_r^2f^2\gamma_0}\frac{\partial^2}{\partial t^2}|\mathbf{A}_\perp|^2. \quad (38)$$

We now assume that the perturbed quantities have the space and time dependence $e^{ikz-i\omega t}$ as in Sec. I. Thus, we obtain the approximate expression

$$n_2^L=\frac{\omega_p^2}{8\pi m v_r^2 f^2\gamma_0}(1+v_0v_r)\left(\frac{\omega^2}{\omega'^2-\omega_p^2/(f\gamma_0^2)}\right)|\mathbf{A}_\perp|^2, \quad (39)$$

where $\omega'=\omega-v_0k$.

In the Gupta-Bleuler theory, the transverse photons are the only degrees of freedom involved in the radiation field. Thus, the longitudinal and scalar photons are not observed as free particles [11]. We can calculate the total photon charge associated to the transverse electromagnetic field as $Q_\perp=-en_2^L$. First, we note that we need to replace $|\mathbf{A}_\perp|^2 \to A_\perp^{\mu +}(z,t)A_\perp^{\mu -}(z,t)$, where $A_\perp^{\mu}(z,t)$ represents the transverse part of the photon field of Eq. (18) ($r=1,2$). In this sense, for a quantum state $|\phi\rangle$, the mean value of the total photon charge $Q_\perp=-e\langle\phi|n_2^L|\phi\rangle=-e\langle n_2^L\rangle$ can be written as

$$Q_\perp=\sum_{r=1,2}\int\frac{-e\hbar\omega_p^2(1+v_0v_r)\omega_r\langle a_r^\dagger(k)a_r(k)\rangle}{16\pi m v_r^2f^2\gamma_0(\omega_r'^2-\omega_p^2/(f\gamma_0^2))}\frac{dk}{(2\pi)^3}. \quad (40)$$

Defining the photon occupation number $n_r(k)=\langle a_r^\dagger(k)a_r(k)\rangle$ for $r=1,2$, the total charge is

$$Q_\perp=\sum_{r=1,2}\int q_r(k,T)n_r(k)\frac{dk}{(2\pi)^3}, \quad (41)$$

where it follows that we can associate to each transverse photon in the plasma a relativistic effective photon charge

$$q_r(k,T)=\frac{-e\hbar\omega_p^2\omega_r(1+v_0v_r)}{16\pi m v_r^2f^2\gamma_0(\omega_r'^2-\omega_p^2/(f\gamma_0^2))} = \frac{-e\hbar\omega_p^2(k^2+\omega_p^2/f)(\sqrt{k^2+\omega_p^2/f}+kv_0)}{16\pi m k^2f^2\gamma_0[(\sqrt{k^2+\omega_p^2/f}-kv_0)^2-\omega_p^2/(f\gamma_0^2)]}. \quad (42)$$

Notice that this effective charge is negative. Then, photons push electrons away from the space occupied by the electromagnetic wave packet. The nonlinear repulsion force effect produced by this equivalent charge corresponds to the radiation pressure of the transverse electromagnetic fields due to the relativistic ponderomotive force [15].

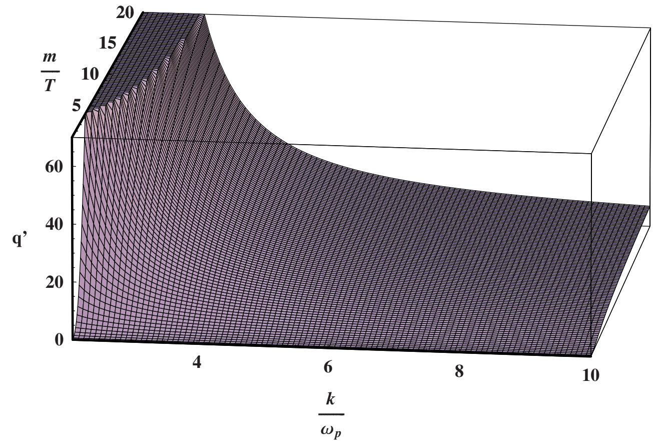

FIG. 2. (Color online) Dependence of the photon charge (42) using $q'\equiv 16\pi m q_r(k,T)/(-e\hbar\omega_p)$ with respect to m/T and k/ω_p. A Lorentz factor $\gamma_0=2.5$ was chosen.

In the cold, nonrelativistic limit regime $v_0=0$ and $T=0$. Besides, we focus only on a time scale faster than the electron plasma oscillation time scale $\omega \gg \omega_p$. Since $\gamma_0 \gtrsim 1$ and $f \gtrsim 1$ for any temperature, we have $\omega \gg \omega_p/(\sqrt{f}\gamma_0)$. Then the density plasma oscillation term in the first line of Eq. (42) can be neglected. Thus, the equivalent photon charge becomes $q_r(k,0)=-e\hbar\omega_p^2/[16\pi m v_r^2\omega_r(k)]$, recovering the result obtained in Ref. [6].

On the other hand, if $T \gg m$ then $f(T)\approx 4T/m$. Hence, the photon charge is

$$q_r(k,T\gg m)=\frac{-e\hbar m\omega_p^2[k^2+m\omega_p^2/(4T)]}{256\pi k^2T^2\gamma_0} \times\frac{[\sqrt{k^2+m\omega_p^2/(4T)}+kv_0]}{[\sqrt{k^2+m\omega_p^2/(4T)}-kv_0]^2-m\omega_p^2/(4T\gamma_0^2)}, \quad (43)$$

which vanishes when the temperature increases.

As discussed in Sec. I, for an ultrarelativistic plasma, where electrons and photons are in thermodynamical equilibrium, $n\sim T^3$, and the temperature dependence of the density must be considered in Eq. (43). However, as long as our isothermal approximation is valid, density and temperature will be decoupled. In any case, we see that the photon charge vanishes for any temperature in the limit $1/\gamma_0 \to 0$.

In Fig. 2, we plot the dependence with temperature T and wave number k of the relativistic photon charge (42) in the regime $\omega \gg \omega_p/(\sqrt{f}\gamma_0)$, where $k/\omega_p>1$. The relativistic factor is $\gamma_0=2.5$.

In order to roughly estimate how relevant are these relativistic calculations in actual physical scenarios, we take the case of an intense laser propagating through a plasma. Taking, for instance, a plasma, at rest, with a number density $n_0 \simeq 10^{20}$ cm^{-3} [16], at a temperature $k_BT \simeq 100$ eV [17], then the effective relativistic photon mass for transverse photons is $M_{1,2} \simeq 7\times 10^{-7} m_e$, and for longitudinal photons is $M_3 \simeq 4\times 10^{-3} m_e$, where m_e is the electron mass. Notice that these results essentially correspond to the nonrelativistic one (since $f\simeq 1$) [6]. The effective mass for transverse photons, in fact, is not very sensitive to relativistic effects. For $f=1.2$ (corresponding to $k_B T\simeq 40$ KeV), $M_{1,2}$ is of the same order of magnitude. For longitudinal photons, instead, $M_3 \simeq 10^{-5} m_e$. Much larger effective masses should be obtained for temperatures such that electrons and photons are in thermal equilibrium, in which case the effective mass will increase linearly with temperature as mentioned at the end of Sec. I. We propose to pursuit this idea elsewhere.

Regarding the equivalent photon charge, for a photon frequency $\omega_r \simeq 10\omega_p$ [6], and $f\simeq 1.2$, and all other parameters as in the previous paragraph, then the photon charge is $q_r \simeq -10^{-9} e$, again about the same order of magnitude as the nonrelativistic result [6]. This charge can be much larger for frequencies near the effective plasma frequency, where the group velocity is small [see Eq. (42)]. For instance, for the same parameters as above, but for $\omega_r = 1.001\omega_p/\sqrt{f}$, then the group velocity is $v_r \simeq 4\times 10^{-2} c$, and the equivalent photon charge is $q_r \simeq 3\times 10^{-3} e$.

III. SUMMARY

We have calculated, by applying second covariant quantization of the electromagnetic field to relativistic thermal plasmas, a relativistic effective mass for transverse and longitudinal photons, and a relativistic equivalent charge for transverse photons. Thermal effects are introduced by means of the magnetofluid unification approach proposed in Ref. [7], which unifies the electromagnetic field and the charged thermal fluid field of the electrons in a single classical field. The relativistic photon mass is a linear result derived from the quantum dispersion relations in analogy with Klein-Gordon fields, whereas the effect of the relativistic photon charge on electrons is analogous to the ponderomotive force due to the transverse electromagnetic field. Both results obtained in this work are the relativistic generalizations to high-energy plasmas of the photon mass and charge obtained in Ref. [6].

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APPENDIX

In this appendix, we derive Eq. (5) for the enthalpy of a relativistic ideal gas. The Hamiltonian of an ideal gas of N identical noninteracting and nondegenerate relativistic particles with mass m, each one with momentum p_i, is

$$\mathcal{H} = \sum_{i=1}^{N} m\sqrt{1+\left(\frac{p_i}{m}\right)^2}. \tag{A1}$$

The partition function is $Z_N(T,V)=(Z_1(T,V))^N/N!$, where

$$Z_1(T,V) = \frac{4\pi V}{(2\pi\hbar)^3}\int_0^\infty dpp^2 \exp\left(-\beta m\sqrt{1+\left(\frac{p}{m}\right)^2}\right). \tag{A.2}$$

Here V is the volume, $\beta=T^{-1}$, T is the temperature and $\hbar$ is the reduced Planck constant.

Setting $p=m\sinh x$ and $u=m\beta$ in Eq. (A2), we obtain

$$Z_1(T,V) = \frac{4\pi V m^3}{(2\pi\hbar)^3}\int_0^\infty dx \cosh x \sinh^2 x \exp(-u\cosh x), \tag{A3}$$

which can be written in terms of the modified Bessel function of order n, K_n, as

$$Z_1(T,V) = 4\pi V\left(\frac{m}{2\pi\hbar}\right)^3 \frac{K_2(u)}{u}. \tag{A4}$$

For $N\gg 1$, the free energy $F(T,V,N)=-T\ln Z_N(T,V)$ is

$$F(T,V,N) = -NT\left\{\ln\left[\frac{4\pi V}{N}\left(\frac{m}{2\pi\hbar}\right)^3\frac{K_2(u)}{u}\right]+1\right\}. \tag{A5}$$

Using the relation $(d/du)K_n(u)=-K_{n-1}(u)-nK_n(u)/u$, the entropy $S=-(\partial F/\partial T)|_{N,V}$ can be written as

$$S = N\ln\left[\frac{4\pi V}{N}\left(\frac{m}{2\pi\hbar}\right)^3\frac{K_2(u)}{u}\right]+4N+\frac{Nm}{T}\frac{K_1(u)}{K_2(u)}. \tag{A6}$$

On the other hand, pressure is given by

$$p=-(\partial F/\partial V)|_{T,N} = nT, \tag{A.7}$$

where $n=N/V$ is the density. Using Eq. (A5), Eq. (A6) and the recursion relation $K_{n-1}(u)=K_{n+1}(u)-2nK_n(u)/u$, the internal energy U and the enthalpy H of this relativistic system are

$$U = F+TS = Nm\frac{K_3(u)}{K_2(u)} - NT, \tag{A8}$$

and

$$H = U+pV = Nm\frac{K_3(u)}{K_2(u)}. \tag{A9}$$

respectively.

Thus the enthalpy density $h=H/V$ can be written as $h=nmf(T)$, where the $f(T)$ function is given by Eq. (5).

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